December 1987 LIDS-P-1722

An operator-theoretic approach to the mixed-sensitivity minimization problem (I)

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December, 1987

Abstract: in this paper we consider the mixed-sensitivity minimization problem (scalar case).It gives rise to the so-called two-block problem on the algebra H^{∞}; we analyze this problem from an operator point of view, using Krein space theory. We obtain a necessary and sufficient condition for the uniqueness of the solution and a parameterization of all solutions in the non-uniqueness case. Moreover, an interpolation interpretation is given for the finite-dimensional case.

(I) This research has been partially supported by the Air Force Office of Scientific Research under Grant N. AFOSR-85-0227

Report Documentation Page	*Form Approved OMB No. 0704-0188*

1. REPORT DATE **DEC 1987**	2. REPORT TYPE	3. DATES COVERED **00-12-1987 to 00-12-1987**

4. TITLE AND SUBTITLE **An operator-theoretic approach to the mixed-sensitivity minimization problem**	5a. CONTRACT NUMBER 5b. GRANT NUMBER 5c. PROGRAM ELEMENT NUMBER
6. AUTHOR(S)	5d. PROJECT NUMBER 5e. TASK NUMBER 5f. WORK UNIT NUMBER
7. PERFORMING ORGANIZATION NAME(S) AND ADDRESS(ES) **Massachusetts Institute of Technology,Laboratory for Information and Decision Systems,77 Massachusetts Avenue,Cambridge,MA,02139-4307**	8. PERFORMING ORGANIZATION REPORT NUMBER
9. SPONSORING/MONITORING AGENCY NAME(S) AND ADDRESS(ES)	10. SPONSOR/MONITOR'S ACRONYM(S) 11. SPONSOR/MONITOR'S REPORT NUMBER(S)

12. DISTRIBUTION/AVAILABILITY STATEMENT

Approved for public release; distribution unlimited

13. SUPPLEMENTARY NOTES

14. ABSTRACT

15. SUBJECT TERMS

16. SECURITY CLASSIFICATION OF:			17. LIMITATION OF ABSTRACT	18. NUMBER OF PAGES	19a. NAME OF RESPONSIBLE PERSON
a. REPORT **unclassified**	b. ABSTRACT **unclassified**	c. THIS PAGE **unclassified**		**32**	

Standard Form 298 (Rev. 8-98)
Prescribed by ANSI Std Z39-18

INTRODUCTION

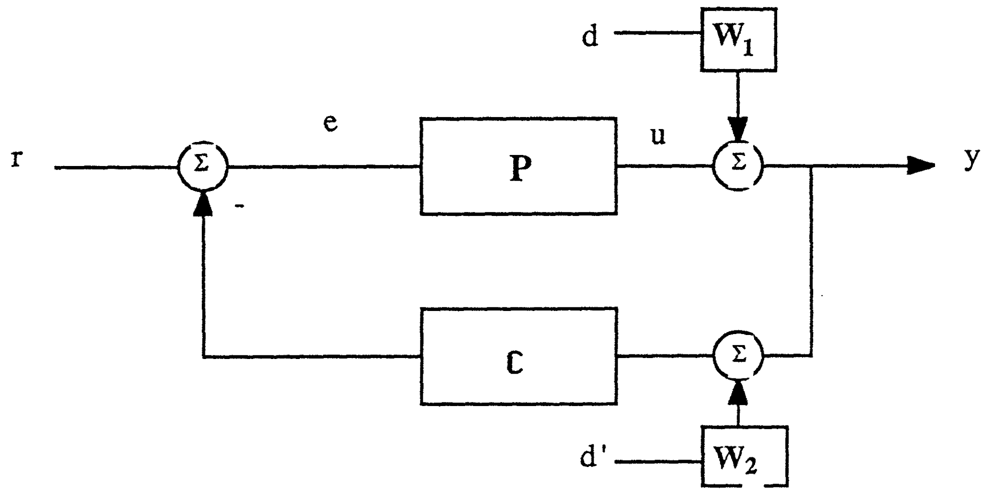
§1 The mixed-sensitivity minimization problem.

The problem we want to study in this paper is a classical one in Control Theory and it is usually known as the mixed-sensitivity minimization problem; it will be precisely stated later in the paragraph.

Throughout this paragraph all linear systems considered will be causal, time invariant, continuos-time, single-input / single-output; moreover no formal distinction will be done between a system and its transfer function.

Let us consider now the following feedback system:

(I)



P is the plant and C is the control system; W_1, W_2 are two weighting functions.

Our goal is to minimize (in some sense) the effect of the disturbances d and d' on the plant P. It is easy to verify that the transfer functions from d to y and from d' to y are, respectively:

$$W_1(1+PC)^{-1} \qquad\qquad W_2PC(1+PC)^{-1}$$

Let $H^\infty(\Pi^r)$ be the Banach algebra of holomorphic, uniformly bounded, complex functions on Π^r (the open right half-plane of $\mathbb{C}$) with the infinity-norm. $H^\infty(\Pi^r)$ may be seen, in a natural way, as a closed subspace of $L^\infty(i\mathbb{R})$, the space of essentially bounded, measurable functions on the imaginary axis (the identification is obtained considering the extension of the holomorphic function to the boundary $i\mathbb{R}$); we will think of a $H^\infty(\Pi^r)$-function in these different ways depending on the context. From a systems point of view $H^\infty(\Pi^r)$ is just the algebra of transfer functions of systems which are linear, causal, time-invariant, continous-time and L^2-stable. Every $H^\infty(\Pi^r)$-function f may be factorized in the following way: $f = f_i f_o$ where $f_i \in H^\infty$ is such that $| f_i | = 1$ almost everywhere on the imaginary axis (it is

said the inner factor of f) and f_o is the outer factor of f. f_i and f_o are uniquely determined up to multiplicative complex units. Throughout this paragraph we set $H^\infty := H^\infty(\Pi^r)$.

Assume that W_1, W_2 belong to H^∞. C is an admissible feedback control if C is causal and if $(1+PC)^{-1}$ and $P(1+PC)^{-1}$ belong to H^∞. So now it is meaningful to state the following H^∞ optimal problem:

$$(1) \qquad \min_{C \text{ adm.}} \left\| \begin{bmatrix} W_1 (1+PC)^{-1} \\ W_2 PC(1+PC)^{-1} \end{bmatrix} \right\|_\infty$$

The function $S := (1+PC)^{-1}$ is the sensitivity function; for $W_2 = 0$, (1) reduces to the classical optimal sensitivity problem which naturally leads to a Nehari problem. The function $PC(1+PC)^{-1}$ is equal to 1-S and is called complementary sensitivity; for this reason (1) is termed the mixed-sensitivity minimization problem.

The problem (1) looks like as a very hard one because it is non-linear in the control and, moreover, the space on which it is defined, is not well characterized. As in the case of the sensitivity problem, (1) may be transformed into a much simpler minimization problem; we are going to show this fact in the next paragraph.

§2 The canonical form of the minimization problem.

In order to transform (1) in the canonical form, it is necessary to place additional hypotheses:

H1) there exists a coprime factorization for P, that is $\exists$ N, D, a, b $\in H^\infty$ s. t.

$$P = ND^{-1} \qquad aN + bD \equiv 1;$$

moreover the outer factors of N and D are invertible in the algebra H^∞.

H2) $$P(a+ib) \to 0 \quad \text{if } a \to +\infty \qquad \forall b$$

H3) W_1, W_2 are outer invertible in H^∞.

H1) and H2) allow us to use Youla parametrization of admissible controls; we have:

$$(2) \qquad C \text{ is admissible} \Leftrightarrow \exists\, Z \in H^\infty \;\; Z \neq N^{-1}b \;\text{ s. t. } C = (a+DZ)(b-NZ)^{-1}$$

The proof of (2) may be found, for example in [Desoer; 1980] or [Francis; 1987].

From (2) we have:

$$S = D(b\text{-}NZ)\,,\; 1\text{-}S = N(a+DZ)$$

so (1) is equivalent to:

(3) $$\operatorname*{Min}_{Z\in H^\infty} \left\| \begin{bmatrix} G_1 \\ G_2 \end{bmatrix} - \begin{bmatrix} H_1 \\ H_2 \end{bmatrix} Z \right\|_\infty$$

where: $$G_1 := W_1Db \qquad H_1 := W_1DN$$ $$G_2 := -W_2Na \qquad H_2 := W_2DN$$

We are going to transform (3) now, following [J.V.1986]. First, we need to remind the concept of spectral factorization. If $A\in H^\infty$ define A^* by $A^*(s) := \overline{A}(-\overline{s})$ $\forall s\in \Pi^l$ (the left open half-plane); clearly A^* admits an L^∞-extension to the immaginary axis and we have $A^*(ix)= \overline{A}(ix)$ $\forall x\in \mathbb{R}$ so that $A^*A = |A|^2$ on $i\mathbb{R}$.

<u>Definition</u> Let $f \in L^\infty$; $f \geq 0$ a.e. We say that there exists a spectral factorization for f $:\Leftrightarrow$ $\exists$ $g\in H^\infty$ such that $g^*g = f$ a.e. on $i\mathbb{R}$.

<u>Proposition</u> (see [Hoffman, 1962]) $f \in L^\infty$; $f \geq 0$ a.e. admits a spectral factorization $\Leftrightarrow$ $\log f \in L^1(d\lambda/(1+t^2))$ where λ is the Lebesgue measure on $i\mathbb{R}$.

<u>Remark</u> If we assume that the spectral factor g is outer than we have the uniqueness of the spectral factorization upto multiplicative complex units.

Set now:

$$T := \begin{bmatrix} G_1 \\ G_2 \end{bmatrix} - \begin{bmatrix} H_1 \\ H_2 \end{bmatrix} Z$$

$$T^*T = G_1{}^*G_1 + G_2{}^*G_2 - (G_1{}^*H_1 + G_2{}^*H_2)Z - (H_1{}^*G_1 + H_2{}^*G_2)Z^* + (H_1{}^*H_1 + H_2{}^*H_2)Z^*Z$$

We have $$H_1{}^*H_1 + H_2{}^*H_2 = D^*DN^*N\,(W_1{}^*W_1 + W_2{}^*W_2)$$

From hypotheses H1) , H3) and the preceding proposition it follows that this function admits a spectral factorization with spectral factor M invertible in H^∞. Now let G be the L^∞-function such that:

$$G^* = M^{-1}(G_1{}^*H_1 + G_2{}^*H_2)$$

Then:

$$T^*T = (G\text{-}MZ)^*(G\text{-}MZ) + (G_1{}^*G_1 + G_2{}^*G_2 - G^*G)$$

It is easily shown that:

$$G_1^*G_1 + G_2^*G_2 - G^*G = W_1^*W_1 W_2^*W_2 (W_1^*W_1 + W_2^*W_2)^{-1}$$

so, by hypothesis H3) and the proposition $\exists F$ spectral outer factor for the above function. So we obtain:

$$T^*T = (G-MZ)^*(G-MZ) + F^*F$$

Therefore problem (3) is equivalent to the following:

$$(4) \qquad \min_{Z\in H^\infty} \left\| \begin{bmatrix} G - Z \\ F \end{bmatrix} \right\|_\infty$$

It is important to observe that the function $F\in H^\infty$ does not depend on the plant P, but only on the two weighting functions W_1, W_2; it is rational if W_1, W_2 are.

We are going to study problem (4) with the assumption that the L^∞-function G is factorizable in the following way: $G = \bar{\psi}W$ where $\psi\in H^\infty$ is inner and $W\in H^\infty$. Looking at the way G is linked to P, W_1, W_2, it is easy to realize that it is not a strong assumption: it is true for example in the case P stable, W_1, W_2 rational functions. Finally, we can state the problem in the following way:

$$(5) \qquad \min_{Z\in H^\infty} \left\| \begin{bmatrix} W - \Psi Z \\ F \end{bmatrix} \right\|_\infty$$

where $\psi\in H^\infty$ is inner and $W\in H^\infty$.

§3 Our approach to the problem

Our approach to problem (5) will be, essentially, operator theoretic; in fact, as in the case of the Nehari problem, operator theory seems to be a very powerful tool to analyze such problems.

In the next five chapters we generalize most of the techniques and the results developed in [Sarason; 1985] for the Nehari problem; we will show how our problem is connected to an extension problem for a given operator on a Hilbert space. A similar approach has already been used to analyze H^∞-problems including problem (5) in [B.H. 1983] and [B.H. 1986]. However we obtain more detailed results, for example in the parametrization of solutions (chapter 5) and in obtaining a necessary and sufficient condition for the uniqueness of the solution (chapter 4). Moreover, in chapter 4 we state one more uniqueness criterion which also gives the form of the solution. Finally, in chapter 6, we give an interpolation interpretation of problem (5) in the finite-dimensional case showing how it generalizes the classical

Nevanlinna-Pick interpolation problem.

The problem (5) is also known as the two-block problem because of the evident two-block structure of the function. In this paper, we treat the scalar case; our approach may be generalized to the matrix case that is to the case in which the two blocks are matrices (and this is done in [B.H. 1983] and [B.H. 1986]), but it does not seem possible to carry out the same analysis as in the scalar case.

The two-block problem is a particular case of the more general four-block problem coming from a general H^∞-control problem (see, for example, [Francis, 1986]). In a forthcoming paper we shall consider such a problem showing how it may be analyzed by the same operator-theoretic techniques.

CHAPTER ONE
Some mathematical preliminaries.

The two main mathematical tools used in this paper are Krein space theory and Hardy space theory; in this chapter we want to remind all the material used in the sequel. In the first paragraph we give a short introduction to Krein spaces following [Sarason; 1985].

§1 Krein spaces

Def 1.1 A **Krein space** is a pair (H,J) where H is a complex Hilbert space and J is a symmetry on H, that is, a self-adjoint unitary operator on H. To eliminate trivial cases we assume that J is different from $\pm I$.

The symmetry J induces an indefinite inner product on H given by (Jx,y) where $x,y \in H$, denoted by $[x,y]$. Obviously $\sigma(J)=\{-1,+1\}$; let us denote by H_+ and H_- the corresponding eigenspaces and by P_+ and P_- the ortogonal projections. Thus $J = P_+ - P_-$ and

$$[x,y] = (P_+x, P_+y) - (P_-x, P_-y).$$

Def 2.1 A vector $x \in H$ is called positive iff $[x,x] \geq 0$. A subspace of H is called positive iff it consists of positive vectors. A positive subspace is said **maximal positive** iff it is not properly contained in another positive subspace. Negative vectors and subspaces are analogously defined.

Prop 3.1 $K \leq H$ is a positive subspace $\Leftrightarrow$ $\exists\ T : D \leq H_+ \rightarrow H_-$ contraction such that $G(T)=K$ (where $G(T)$ is the graph of T). Moreover K is maximal positive $\Leftrightarrow$ $D= H_+$. The

operator T is said to be the angular operator of K.

Def 4.1 A positive subspace is said to be **uniformly positive** iff the norm of its angular operator is less than one.

Using the indefinite inner product [,] one can define the J-ortogonality between vectors and subspaces (indicated by $[\perp]$), the J-adjoint of an operator (indicated by a $^{[*]}$) and so on.

Def 5.1 $K \le H$ is called **regular** iff $\exists\ M_+ \le H$ uniformly positive and $M_- \le H$ uniformly negative, J- ortogonal, such that $K = M_+ \oplus M_-$.

Regular subspaces have nice properties as the following:

Obs 6.1 Let $K \le H$ regular and D a linear manifold in K; then D is dense in K if and o
if no nonzero vector in K is J-ortogonal to it.

Prop 7.1 $K \le H$ is regular $\Leftrightarrow H = K \oplus K^{[\perp]}$. In particular K is regular if and only if $K^{[\perp]}$ is regular.

Example The simplest example of a Krein space is the following: let us consider the finite-dimensional Hilbert space $\mathbb{C}^m \oplus \mathbb{C}^n$; it is a Krein space with the isometry $J_{m,n}$ given by $J_{m,n}(x,y) := (x,-y)$.

§2 Hardy spaces on the unit disk.

In this paragraph we want to recall the main facts regarding Hardy space theory; we essentially follow [Hoffman,1962].

Set the following notation: Δ is the unit open disk in the complex plane; $\mathbb{T} := \partial\Delta$. If $f \in Hol(\Delta, C)$ then f_r indicates the function $\theta \to f(re^{i\theta})$; $\| \ \|_p$ is the canonical norm on $L^p(\mathbb{T}, \mathbb{C})$ where $p \ge 1$.

Def 8.1

$$H^p(\Delta) := \{\ f \in Hol(\Delta, C) \text{ s.t. } \sup\{\| f_r \|_p \mid r \in (0,1)\} < +\infty\}$$

It is a Banach space with the norm: $\| f \|_p := \sup\{\| f_r \|_p \mid r \in (0,1)\}$

Prop 9.1 Let $f \in H^p(\Delta)$ then $\exists\ f' \in L^p(\mathbb{T}, \mathbb{C})$ such that $f_r \to f'$ a.e.
Moreover the map $f \to f'$ is an isometry between $H^p(\Delta)$ and $L^p(\mathbb{T}, \mathbb{C})$

From now on we will simply indicate by H^p and L^p the spaces, respectively, $H^p(\Delta)$ and $L^p(\mathbb{T},\mathbb{C})$.

Remark From the preceding proposition we deduce that H^p may be identified with a closed subspace of L^p. This identification will be used throughout this paper; depending on the case, a H^p-function will be thought as an analytic function on Δ or as an L^p-function on $\mathbb{T}$.

Def 10.1 Let $f \in H^p$.

1) f is said **inner** $:\Leftrightarrow |f(e^{i\theta})| = 1$, θ-a.e. (in particular $f \in H^\infty$).

2) f is said **outer** $:\Leftrightarrow \mathrm{clos}\{e^{in\theta}f \mid n \geq 0\} = H^p$

Prop 11.1 **(inner-outer factorization)** Let $f \in H^p$.

Then $\exists\ \phi \in H^\infty$ inner, $\exists\ g \in H^p$ outer s.t. $f = \phi g$.

Moreover, the factorization is unique up to multiplicative complex units.

The two Hardy spaces which will be mainly used in the sequel, are: H^∞ which is a Banach algebra, and H^2 which is, in a natural way, a Hilbert space. H^2 may be seen as the subspace of L^2 spanned by the functions $\{e^{in\theta} \mid n \geq 0\}$; on L^2 the unitary operator T which acts as $Tf := e^{i\theta}f$ is called the **right bilateral shift**; H^2 is a closed invariant subspace for T and the restriction S of T to H^2 is said the **right unilateral shift** (it is still an isometry but no longer unitary). The following result is fundamental:

Prop 12.1 **(Beurling-Lax)** Let $K \leq H^2$ be a closed, non-zero, S-invariant subspace.

Then $\exists\ \psi \in H^\infty$ inner s.t. $K = \psi H^2$

Moreover, the representation is unique up to multiplicative complex units.

We finish this paragraph by recalling some other useful definitions. Let us indicate by P_+ the projection on H^2 and by P_- the projection on $(H^2)^\perp$

Def 13.1 Let $W \in L^\infty$. We define:

$\mathbf{M}_W: L^2 \to L^2$	$\mathbf{M}_W(f) := Wf$	**Laurent operator**
$\mathcal{H}_W: H^2 \to (H^2)^\perp$	$\mathcal{H}_W(f) := P_-(Wf)$	**Hankel operator**
$\mathbf{T}_W: H^2 \to H^2$	$\mathbf{T}_W(f) := P_+(Wf)$	**Toeplitz operator**

W is said to be the symbol of the corresponding operator.

Remark: While the symbol is unique for the Laurent and Toeplitz operator (in the sense that the two maps $W \to \mathbf{M}_W$ and $W \to \mathbf{T}_W$ are injective) and $\| \mathbf{M}_W \| = \| \mathbf{T}_W \| = \| W \|_\infty$, this is not the case for the Hankel operator. In fact $\mathcal{H}_W$ does not change if we modify the symbol W by adding a H^∞-function; moreover we have only $\| \mathcal{H}_W \| \leq \| W \|_\infty$ and, obviously, the inequality may be strict. It is an important theorem (known as the **Nehari theorem**) the fact that every Hankel operator has at least one symbol W', called **minimal symbol,** such that $\|\mathcal{H}_{W'} \| = \| W' \|_\infty$.

In paragraph 1 of the introduction we defined the algebra $H^\infty(\Pi^r)$ on which we have stated our minimization problem. There is a nice isometric isomorphism between the two algebras $H^\infty(\Pi^r)$ and $H^\infty(\Delta)$ induced by the well-known Cayley transform:

$$f \in H^\infty(\Pi^r) \to f' \in H^\infty(\Delta)$$
$$f'(z) := f((1+z)(1-z)^{-1})$$

It is thus equivalent to study our problem (5) on $H^\infty(\Delta)$ instead of on $H^\infty(\Pi^r)$. The theory on the unit disk is simpler, at least from a formal point of view, and so, from now on, we will work on the unit disk Δ.

CHAPTER TWO
Statement of the problem in the operator theory context.

§1 Some preliminaries

Let us begin by writing down again the H^∞ optimal problem:

$$(1) \qquad \min_{Z \in H^\infty} \left\| \begin{bmatrix} W - \Psi Z \\ F \end{bmatrix} \right\|_\infty = \varepsilon$$

Let us consider now the following operator:

$$(2) \qquad \mathcal{A} : H^2 \to (H^2)^\perp \oplus H^2$$

given by:

$$(3) \qquad \mathcal{A}\phi := (\mathcal{H}_{\bar{\psi}W}\phi \, , \, \mathbf{T}_F\phi)$$

Clearly, the operator $\mathcal{A}$ remains unchanged if we modify $\bar{\psi}W$ by adding an H^∞ function.

The pairs:

$$\begin{bmatrix} \overline{\Psi}W - Z \\ F \end{bmatrix}$$

where $Z \in H^\infty$, are said to be symbols of $\mathcal{A}$.

We have:

$$\|\mathcal{A}\| \leq \left\| \begin{bmatrix} \overline{\Psi}W - Z \\ F \end{bmatrix} \right\|_\infty = \left\| \begin{bmatrix} W - \Psi Z \\ F \end{bmatrix} \right\|_\infty$$

So:

$$\|\mathcal{A}\| \leq \varepsilon$$

The operator $\mathcal{A}$ behaves as a Hankel operator; the preceding inequality is, in fact, an equality so that the solutions of (1) are just the "minimal" symbols of $\mathcal{A}$.

The problem of finding the minimal symbols of $\mathcal{A}$ is known as the extension problem for the operator $\mathcal{A}$; the reason for this is explained in what follows. Every symbol of $\mathcal{A}$ naturally induces a multiplicative operator:

$$\begin{bmatrix} \overline{\Psi}W - Z \\ F \end{bmatrix} \longrightarrow \qquad \begin{aligned} &\mathbf{M} : H^2 \to L^2 \oplus H^2 \\ &\mathbf{M}_\phi := [(\overline{\Psi}W - Z)\phi , F\phi] \end{aligned}$$

$\mathbf{M}$ is said to be a dilation of $\mathcal{A}$ on the space $L^2 \oplus H^2$. Let us note that :

$$\|\mathbf{M}\| = \left\| \begin{bmatrix} W - \Psi Z \\ F \end{bmatrix} \right\|_\infty$$

Such multiplicative operators are precisely the operators from H^2 to $L^2 \oplus H^2$ commuting with the right shift. So, the problem of finding the minimal symbols of $\mathcal{A}$ is the same as the problem of finding the minimal dilations of $\mathcal{A}$ on $L^2 \oplus H^2$ commuting with the right shift. The reason for which this is said to be an extension problem and not, merely, a dilation problem is that the adjoint of each dilation of $\mathcal{A}$ is a real extension of $\mathcal{A}^*$. We now start to study the extension problem for the operator $\mathcal{A}$ using Krein spaces theory.

§2 The optimal problem in the Krein spaces context

We introduce now the following Krein space:

(4) $$\mathcal{H} := L^2 \oplus H^2 \oplus H^2$$

with the indefinite scalar product given by:

(5) $$[(f_1, f_2, f_3) , (g_1, g_2, g_3)] := < f_1, g_1>_{L^2} + < f_2, g_2>_{H^2} - < f_3, g_3>_{H^2}$$

Let J be the corresponding symmetry. On $\mathcal{H}$ we may consider the right shift S given by:

$$S: \mathcal{H} \to \mathcal{H} \qquad Sf := e^{i\theta}f$$

If we indicate by S_{L^2} the bilateral right shift on L^2 and by S_{H^2} the unilateral right shift on H^2, we have:

$$S\,(f_1, f_2, f_3) = (\, S_{L^2} f_1 \,, S_{H^2} f_2 \,, S_{H^2} f_3 \,)$$

Let us recall that:

$$\mathcal{A} : H^2 \to (H^2)^{\perp} \oplus H^2 \qquad \mathcal{A}\phi := (\, \mathcal{H}_{\bar{\psi}W}\phi \,,\ T_F\phi \,)$$

so:

$$\mathcal{A}^*: (H^2)^{\perp} \oplus H^2 \to H^2 \qquad \mathcal{A}^*(\,\phi_1, \phi_2) := \mathcal{H}^*_{\bar{\psi}W}\,\phi_1 + T^*_F\,\phi_2$$

Now we have:

$$(6) \qquad \mathcal{H}^*_{\bar{\psi}W}\, S_{L^2}{}^*\,|_{(H^2)^{\perp}} = S_{H^2}{}^*\,\mathcal{H}^*_{\bar{\psi}W} \qquad T^*_F\, S_{H^2}{}^* = S_{H^2}{}^*\, T^*_F$$

so that:

$$\mathcal{A}^*(\, S_{L^2}{}^*\,|_{(H^2)^{\perp}}\phi_1 \,,\ S_{H^2}{}^*\phi_2 \,) = S_{H^2}{}^*\mathcal{A}^*(\,\phi_1, \phi_2)$$

from which we easily derive that $G(\mathcal{A}^*)$, the graph of $\mathcal{A}^*$, seen as a subspace of $\mathcal{H}$, is S^*-invariant.

Let us state now the following fundamental:

<u>Theorem 1.2</u> $\qquad \|\mathcal{A}\| < 1 \quad \Rightarrow \quad \exists\, Z \in H^{\infty} \quad$ s. t.

$$\left\| \begin{bmatrix} \bar{\Psi}W - Z \\ F \end{bmatrix} \right\|_{\infty} \leq 1$$

that is $\mathcal{A}$ has a symbol whose norm is not greater than one.

<u>Proof</u>

We follow [Sarason;1985], slightly modifying the proof of the corresponding theorem regarding the Nehari problem.

We have already seen that $G(\mathcal{A}^*) \leq H$ is S^*-invariant; moreover, because of the

assumption on the norm of $\mathcal{A}$, it is uniformly positive. To find symbols of $\mathcal{A}$ whose norm is not greater than one is equivalent to find extensions of the operator $\mathcal{A}^*$ to the domain $L^2 \oplus H^2$ whose operator norm is not greater than one and whose graph is S^*-invariant. So, it is equivalent to find maximal positive, S^*-invariant subspaces of $\mathcal{H}$, containing $G(\mathcal{A}^*)$.

Let us set $N := G(\mathcal{A}^*)^{[\perp]}$. S is an isometry and also a J-isometry (infact we have $S^* = S^{[*]}$); this implies that SN is a regular subspace of N. It is easy to verify the following inductive formula: $N = L + SL + ... + S^{n-1}L + S^nN$, where $L = N \cap (SN)^{[\perp]}$ is a regular subspace and where all the sums are J-orthogonal. So we have that $\operatorname{span}\{S^kL \mid k \geq 0\}^{[\perp]} = \cap\{S^nN \mid n \geq 1\}$. On the other hand $\cap\{S^nN \mid n \geq 1\} \leq \cap\{S^n\mathcal{H} \mid n \geq 1\} = L^2 \oplus \{0\} \oplus \{0\}$ and $\cap\{S^nN \mid n \geq 1\} \leq G(\mathcal{A}^*)^{[\perp]}$; so $\cap\{S^nN \mid n \geq 1\} \leq H^2 \oplus \{0\} \oplus \{0\}$ and it is a reducing subspace for S. By virtue of the Beurling-Lax theorem we have that $\cap\{S^nN \mid n \geq 1\} = \{0\}$. Being N regular, we have that $N = \operatorname{span}\{S^kL \mid k \geq 0\}$.

Let us observe now that L is neither positive, nor negative; in fact: L positive $\Rightarrow$ N positive $\Rightarrow$ $\mathcal{H}$ positive which is absurd; L negative $\Rightarrow$ N negative and this is not possible because N contains $H^2 \oplus \{0\} \oplus \{0\}$ which is uniformly positive. So there exists $x_1 \in L$: $[x_1, x_1] = 1$.

Let us set $N_+ := \operatorname{span}\{ S^k x_1 \mid k \in \mathbb{N}\}$ and let us consider the S^*-invariant subspace $G(\mathcal{A}^*) + N_+$: it is maximal positive. It is obviously positive being the J-orthogonal sum of positive subspace. The only thing we have to show is that $P(G(\mathcal{A}^*) + N_+) = L^2 \oplus H^2$ where $P := P_{L^2 \oplus H^2 \oplus \{0\}}$. We have: $S^*x_1 [\perp] N \Rightarrow Px_1 \in ((H^2)^\perp \oplus C) \oplus H^2$; on the other hand $(H^2)^\perp \oplus H^2 \leq PG(\mathcal{A}^*) \neq P(G(\mathcal{A}^*) + \{x_1\})$. So $P(G(\mathcal{A}^*) + \{x_1\}) = ((H^2)^\perp \oplus C) \oplus H^2$; by induction we obtain the result. We have found a maximal-positive subspace of $\mathcal{H}$ which is S^*-invariant and contains $G(\mathcal{A}^*)$; therefore, the proof is complete. Q.E.D.

<u>Theorem 2.2</u> $\qquad \|\mathcal{A}\| = 1 \quad \Rightarrow \quad \exists\, Z \in H^\infty \quad$ s. t.

$$\left\| \begin{bmatrix} \bar{\Psi} W - Z \\ F \end{bmatrix} \right\|_\infty = 1$$

<u>Proof</u>

Let us consider $\mathcal{A}_\varepsilon := (1-\varepsilon)\mathcal{A}$ $\varepsilon \in (0,1)$. $\|\mathcal{A}_\varepsilon\| \leq 1$. So, by the theorem 1.2 $\exists\, Z_\varepsilon \in H^\infty$:

$$1 \leq \left\| \begin{bmatrix} \bar{\Psi} W - Z_\varepsilon \\ F \end{bmatrix} \right\|_\infty \leq (1-\varepsilon)^{-1}$$

So:

$$\lim_{\varepsilon\to 0} \left\| \begin{bmatrix} \bar{\Psi}W - Z_\varepsilon \\ F \end{bmatrix} \right\|_\infty = 1$$

By a standard compactness argument in the weak-* topology of $L^\infty \oplus H^\infty$ we find $Z \in H^\infty$ s. t. :

$$\left\| \begin{bmatrix} \bar{\Psi}W - Z \\ F \end{bmatrix} \right\|_\infty = 1$$

Q.E.D.

CHAPTER THREE
The parameterization of the symbols in $B(H^\infty)$

§1 The parameterization

The next result we want to obtain is a parameterization of all the symbols of $\mathcal{A}$ having a prescribed norm. Precisely, given $\mathcal{A}$ such that $\|\mathcal{A}\| <1$, we will describe all its symbols whose infinity-norm is not greater than one.

Let us begin with a further analysis of the subspace L introduced in the proof of the theorem 1.2. As we said before, it is regular, so it may be written as the J-orthogonal sum of a uniformly positive subspace L^+ and a uniformly negative one L^-. We observed before that $\dim L^+ \geq 1$; $\dim L^- \geq 1$. In fact:

<u>Lemma 1.3</u> $\dim L^+ = \dim L^- = 1$

<u>Proof</u>

It is obvious that $\dim L^+ = 1$ because, otherwise, $G(\mathcal{A}^*) + N_+$ would not be maximal positive.

$\dim L^- \geq 1 \Rightarrow \exists\, x_2 \in L$ s. t. $[\, x_2 , x_2 \,] = -1$ and $[\, x_1 , x_2 \,] = 0$

Let us consider the Krein space $(\mathbb{C}^3, J_{2,1})$, as defined in the last example of introduction (§1); let $[\ ,\ ]_{2,1}$ be the corresponding indefinite inner product. Then:

$$[\, x_1 , S^n x_1 \,] = \int_0^{2\pi} [\, x_1(e^{i\theta}) , x_1(e^{i\theta}) \,]_{2,1}\, e^{-in\theta} d\theta \qquad \forall n$$

and this implies that the function $[\, x_1(e^{i\theta}), x_1(e^{i\theta}) \,]_{2,1}$ has the same Fourier series as the constant function 1, and thus it is equal to 1 almost everywhere. The same argument may be carried out for the other two orthogonality relations so, finally, we obtain:

(1) $$[\, x_1(e^{i\theta}), x_1(e^{i\theta}) \,]_{2,1} = 1 = - [\, x_2(e^{i\theta}), x_2(e^{i\theta}) \,]_{2,1} \quad \text{a.e.}$$
$$[\, x_1(e^{i\theta}), x_2(e^{i\theta}) \,]_{2,1} = 0 \quad \text{a.e.}$$

$\dim L^- > 1 \Rightarrow \exists\, x_3 \in L$ s. t. $[\, x_3 , x_3 \,] = -1$ and $[\, x_2 , x_3 \,] = 0$
so we also would have:

$$[\, x_3(e^{i\theta}), x_3(e^{i\theta}) \,]_{2,1} = -1 \qquad [\, x_2(e^{i\theta}), x_3(e^{i\theta}) \,]_{2,1} = 0 \quad \text{a.e.}$$

So there should exist $\theta_o \in [0, 2\pi]$ s.t.

$$[\, x_2(e^{i\theta_o}), x_2(e^{i\theta_o}) \,]_{2,1} = [\, x_3(e^{i\theta_o}), x_3(e^{i\theta_o}) \,]_{2,1} = -1$$
$$[\, x_2(e^{i\theta_o}), x_3(e^{i\theta_o}) \,]_{2,1} = 0$$

and this is absurd because $(\mathbb{C}^3, J_{2,1})$ does not have negative subspaces whose dimension is greater than one. Therefore, $\dim L^- = \dim L^+ = 1$ and $\{x_1, x_2\}$ is a base of L. Q.E.D.

We may write:

$$x_1 = p_1 \oplus q_1 \oplus r_1 \qquad p_i \in L^2 \;\; q_i, r_i \in H^2$$
$$x_2 = p_2 \oplus q_2 \oplus r_2$$

Let us consider the matrix:

(2) $$U := \begin{bmatrix} p_1 & p_2 \\ q_1 & q_2 \\ r_1 & r_2 \end{bmatrix}$$

Obs.2.3 If we fix θ, $U(\theta)$ may be thought as a linear map from $\mathbb{C}^2$ to $\mathbb{C}^3$. Because of the pointwise relations (1) we have that:

(3) $$[\, U(\theta)v_1 , U(\theta)v_2]_{2,1} = [\, v_1, v_2]_{1,1} \quad \forall\, v_1, v_2 \in \mathbb{C}^2.$$

Lemma 3.3 r_2 is outer

Proof

The details of the proof may be found in [Sarason; 1985] page 304; here we only give a sketch of it.

$N_- := [\, G(\mathcal{A}^*) + N_+ \,]^{[\perp]}$ is S-invariant and maximal-negative; it is easy to see that it may be represented in the following way: $N_- = \{hx_3 \mid h \in H^\infty\}$ where x_3 is a suitable vector of $\mathcal{H}$. So the last component of x_3 is necessarily outer and the proof ends by showing that $x_2 = h_o x_3$ where $h_o \in H^\infty$ outer. Q.E.D.

As we said at the beginning of this chapter, our goal is to classify all the S^*-invariant, maximal positive subspaces of $\mathcal{H}$, containing $G(\mathcal{A}^*)$; but this is equivalent to classifying all the S-invariant, maximal negative subspaces of $\mathcal{H}$, contained in the space N_- introduced in the preceding lemma. Such subspaces are just the graphs of the multiplicative operators whose inducing functions are the symbols for $\mathcal{A}$ having L^∞-norm of at most 1.

Let us note now that the matrix U may be thought as a linear map between $H^\infty \oplus H^\infty$ and N; moreover the Hilbert space $H^2 \oplus H^2$ may be seen as a Krein space with the indefinite product induced by that of $(\mathbb{C}^2, J_{1,1})$.The next proposition gives a first result about the parameterization.

<u>Prop.4.3</u> Let $N'' \leq H^2 \oplus H^2$ be S-invariant, maximal-negative. Then:
$N' := \mathrm{clos}\, U(N'' \cap H^\infty \oplus H^\infty) \leq \mathcal{H}$ is S-invariant, maximal-negative, contained in N.

<u>Proof.</u>

N' is clearly S-invariant and contained in N; moreover, it is negative due to the relations in (3). It remains to be shown that is maximal-negative. It may be represented in the following way:

$$\exists\, \psi \in B(H^\infty) \text{ s.t. } \quad N'' = \{\psi h \oplus h \mid h \in H^2\}$$

So: $$N' = \mathrm{clos}\, \{(p_1\psi + p_2)h \oplus (q_1\psi + q_2)h \oplus (r_1\psi + r_2)h \mid h \in H^2\}$$

Let us note that $r_1\psi + r_2 \in P_{\{0\}\oplus\{0\}\oplus H^2} N'$. Therefore the proof is complete if we show that $r_1\psi + r_2$ is outer. We have: $r_1\psi + r_2 = r_2(1 + r_2^{-1} r_1 \psi)$; using again (3) we can see that $r_2^{-1} r_1$ is in $B(H^\infty)$ so that $1 + r_2^{-1} r_1 \psi$, being the sum of 1 and of a function in $B(H^\infty)$, is outer. By the lemma 3.3 the proof is complete. Q.E.D.

<u>Obs.5.3</u> The angular operator associated to the subspace N' is the multiplicative operator induced by the pair:

$$(4) \qquad \left((p_1\psi + p_2)(r_1\psi + r_2)^{-1} ,\ (q_1\psi + q_2)(r_1\psi + r_2)^{-1} \right)$$

By the preceding proposition all these are symbols for $\mathcal{A}$; so necessarily:

$$(q_1\psi + q_2)(r_1\psi + r_2)^{-1} = F \quad \forall \psi \in B(H^\infty) \quad \Rightarrow$$

(5) $$q_1 = Fr_1 \quad q_2 = Fr_2$$

We will show now that (4) gives a complete parameterization of all the symbols of $\mathcal{A}$. Let us set:

$$U' := \begin{bmatrix} p_1 & p_2 \\ r_1 & r_2 \end{bmatrix}$$

Let us set the following notation:

(6) $$U'^{t}(\psi, 1) = (p_1\psi + p_2)(r_1\psi + r_2)^{-1}$$

$U'^{t}(1, 1)$, $U'^{t}(0, 1)$ are symbols for $\mathcal{H}_{\bar{\psi}W}$ (by prop.4.3) so that: $U'^{t}(1,1) - U'^{t}(0,1) \in H^\infty$; by a simple calculation we thus derive: $r_2^{-1}(r_1 + r_2)^{-1} \det U' \in H^\infty \Rightarrow \det U' \in H^1$.

Lemma 6.3 $\det U' \in H^\infty$ and it is outer.

Proof

From relations (3) and (5) we easily obtain:

(7) $$U'^{*}\begin{bmatrix} 1 & 0 \\ 0 & -(1-|F|^2) \end{bmatrix} U' = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \Rightarrow$$

(8) $$|\det U'|^2 = (1-|F|^2)^{-1} \text{ a.e. on } \partial\Delta$$

This implies that det U' belongs to H^∞ (we use the fact $\| F \|_\infty \le \| \mathcal{A} \| < 1$) and its outer factor is uniquely determined by the spectral factorization of $(1-|F|^2)^{-1}$.

Let us show now that det U' is outer; it is equivalent to showing that $((\det U')H^2)^\perp$ is trivial. Let $h \in ((\det U')H^2)^\perp$ and let $x = P_H(h \oplus \{0\} \oplus r_2^{-1}p_2 h)$. It is a mere matter of calculation to show that: $x \,[\perp]\, S^n x_1$, $x \,[\perp]\, S^n x_2$ $\forall n \ge 0$; therefore $x \,[\perp]\, N \Rightarrow x \in G(\mathcal{A}^*) \Rightarrow h \in (H^2)^\perp$ $\Rightarrow h = 0$. Q.E.D.

Let us set $d := \det U'$. By (8) we have that $|d|^2 = (1-|F|^2)^{-1}$. So d is exactly the outer factor of the spectral factorization of $(1-|F|^2)^{-1}$ unique up to multiplicative complex units. Let us observe that d is a unit in the algebra H^∞.

Obs.7.3 By manipulating relation (7) established in the preceding lemma we have:

$$(9) \qquad p_1 = \overline{r_2}\ \overline{d}^{-1} \quad p_2 = \overline{r_1}\ \overline{d}^{-1}$$

We may now state the fundamental result of this chapter:

Theorem 8.3 $\quad {}^t(\phi, F)$ is a symbol of $\mathcal{A}$: $\| {}^t(\phi, F) \| \leq 1 \Leftrightarrow$
$\exists \psi \in B(H^\infty) : \ \phi = U'\, {}^t(\psi, 1) \ = \ (p_1\psi + p_2)(r_1\psi + r_2)^{-1}$

Proof.

($\Leftarrow$) has been already proved: it is contained in Prop.4.3.

Let us prove now ($\Rightarrow$). Let ${}^t(\phi, F)$ be a symbol of $\mathcal{A}$: $\| {}^t(\phi, F) \| \leq 1$. U' is invertible a.e. ; let us set $(\psi_1, \psi_2) = U'^{-1}\, {}^t(\phi, 1)$. We have:

$$(10) \qquad U'\, {}^t(\psi_1\psi_2^{-1}, 1) \ = \phi \text{ in the sense of (6)} \Rightarrow$$
$$U''(\psi_1\psi_2^{-1}, 1) \ = \psi_2^{-1}\, {}^t(\phi, F, 1).$$

From the preceding relation, using (3), we obtain $\| \psi_1\psi_2^{-1} \|_\infty \leq 1$.

We complete the proof showing that $\psi := \psi_1\psi_2^{-1} \in H^\infty$. From (9) we have that : $r_2^{-1}\, r_1 d^{-1}$ is a symbol of $\mathcal{H}_{\overline{\psi}W}$ and so $\phi - r_2^{-1}\ \overline{r_1}\ \overline{d}^{-1} \in H^\infty$. From (10) we have that:

$$(11) \qquad \psi = r_2\,(\phi - r_2^{-1}\ \overline{r_1}\ \overline{d}^{-1})(-r_1\phi + \overline{r_2}\ \overline{d}^{-1})^{-1}$$

We may observe that $r_2\,(\phi - r_2^{-1}\ \overline{r_1}\ \overline{d}^{-1})$ belongs to H^2. On the other hand from (9) we derive:

$$(12) \qquad |\, r_2 |^2\ \overline{d}^{-1} - |\, r_1 |^2\ \overline{d}^{-1} = d \ \Rightarrow$$
$$\Rightarrow \qquad \overline{r_1}\ \overline{d}^{-1} = r_1\,(\phi - r_2^{-1}\ \overline{r_1}\ \overline{d}^{-1}) + d r_2^{-1} \in H^2.$$

So $-r_1\phi + \overline{r_2}\ \overline{d}^{-1} \in H^2$. On the other hand: $-r_1\phi + \overline{r_2}\ \overline{d}^{-1} = r_2^{-1} d(\, |\, r_2 |^2 |\, d\, |^{-2} - r_1 r_2 d^{-1}\phi)$ $\Rightarrow |\, r_2 |^2 |\, d\, |^{-2} - r_1 r_2 d^{-1}\, \phi \in H^1$ and, from (12), its real part is always positive; so it is outer. We deduce that ψ is analytic on Δ and so it belongs to H^∞. Q.E.D.

§2 The construction of the matrix U

In this paragraph we want to give an explicit expression of the elements of the matrix U in terms of the operator $\mathcal{A}$.

Obs.9.3 It is obvious that the matrix U is not uniquely determined because the two

vectors x_1 and x_2 are not unique.

Prop.10.3 A possible choice of the elements of U is the following:

$$p_2 = \| (\mathbf{I} - \mathcal{A}^*\mathcal{A})^{-1/2}\mathbf{1}\|_2^{-1} \mathcal{H}_{\bar{\psi}W} (\mathbf{I} - \mathcal{A}^*\mathcal{A})^{-1}\mathbf{1}$$
$$r_2 = \| (\mathbf{I} - \mathcal{A}^*\mathcal{A})^{-1/2}\mathbf{1}\|_2^{-1}(\mathbf{I} - \mathcal{A}^*\mathcal{A})^{-1}\mathbf{1}$$

the other elements of U are linked to p_2 and r_2 by relations (5) and (9).

Proof.

It is easy to see that: $N = H^2\oplus\{0\}\oplus\{0\} \bigoplus G(\mathcal{A})$. On the other hand $L \leq N$ and $S^*L\leq G(\mathcal{A}^*)$, so we have: $L \leq \mathbb{C} \oplus\{0\}\oplus\{0\} \bigoplus G(\mathcal{A})$. Let $x\in L$, then $\exists\ \xi\in H^2$, $\exists\ \alpha\in \mathbb{C}$ s.t.

$$x = (\mathcal{H}_{\bar{\psi}W}\,\xi + \alpha\, , \mathbf{T}_F\,\xi\, , \xi\,) \tag{13}$$

$S^*x\in G(\mathcal{A}^*) \Rightarrow \exists\ \eta_1\in (H^2)^\perp, \eta_2\in H^2$ s.t. $S^*x = \eta_1\oplus\eta_2\oplus\mathcal{H}^*_{\bar{\psi}W}\,\eta_1 + \mathbf{T}^*_F\,\eta_2$. So we have:

$$S_{L^2}^*\ (\mathcal{H}_{\bar{\psi}W}\ \xi + \alpha\ ) = \eta_1$$
$$S_{H^2}^*\ \mathbf{T}_F\,\xi = \eta_2$$
$$S_{H^2}^*\ \xi = \mathcal{H}^*_{\bar{\psi}W}\,\eta_1 + \mathbf{T}^*_F\,\eta_2.$$

Applying $\mathcal{H}^*_{\bar{\psi}W}$ to the first relation, $\mathbf{T}^*_F$ to the second and then summing, we obtain that there exists $\beta\in \mathbb{C}$ s.t.

$$\xi = \alpha(\mathbf{I} - \mathcal{A}^*\mathcal{A})^{-1}\, S_{H^2}\mathcal{H}^*_{\bar{\psi}W}\, S_{L^2}^*\mathbf{1} + \beta(\mathbf{I} - \mathcal{A}^*\mathcal{A})^{-1}\mathbf{1} \tag{14}$$

This is a necessary condition on the pair (ξ, α) so that a vector x as defined in (13) belongs to L but the argument is clearly reversible so that the condition (14) is also sufficient.

If we put $\alpha = 0$ in (14) we obtain $\xi = \beta(\mathbf{I} - \mathcal{A}^*\mathcal{A})^{-1}\mathbf{1}$ and the corresponding vector is:

$$(\beta\mathcal{H}^*_{\bar{\psi}W}(\mathbf{I} - \mathcal{A}^*\mathcal{A})^{-1}\mathbf{1}\, , \beta\mathbf{T}_F(\mathbf{I} - \mathcal{A}^*\mathcal{A})^{-1}\mathbf{1}, \beta(\mathbf{I} - \mathcal{A}^*\mathcal{A})^{-1})$$

$\forall\beta \neq 0$ which is strictly negative and so it is a possible choice for x_2. β will be determined by the condition $[x_2, x_2] = -1$; we obtain $\beta = \| (\mathbf{I} - \mathcal{A}^*\mathcal{A})^{-1/2}\mathbf{1}\|_2^{-1}$. Remembering that p_2 and r_2 are,respectively, the first and the third component of x_2, the proof is complete. Q.E.D.

CHAPTER FOUR
Some results about the uniqueness of the solution

§1 A generalization of Krein's uniqueness condition.

From now on we will suppose $\| \mathcal{A} \| = 1$. Let us consider $\mathcal{A}_\varepsilon := (1-\varepsilon)\mathcal{A}$ $\varepsilon\in(0,1)$; $\|\mathcal{A}_\varepsilon\|<1$. By the result obtained in the last chapter we have that the symbols of $\mathcal{A}_\varepsilon$ in $B(H^\infty)$ are parameterized by the mean of a given matrix U'_ε:

$$U'_\varepsilon := \begin{bmatrix} p_{1\varepsilon} & p_{2\varepsilon} \\ r_{1\varepsilon} & r_{2\varepsilon} \end{bmatrix}$$

where:

$$(1) \qquad \begin{cases} r_{2\varepsilon} = \| (\mathbf{I} - \mathcal{A}_\varepsilon{}^*\mathcal{A}_\varepsilon)^{-1/2}\mathbf{1}\|_2{}^{-1}(\mathbf{I} - \mathcal{A}_\varepsilon{}^*\mathcal{A}_\varepsilon)^{-1}\mathbf{1} \\ p_{2\varepsilon} = \mathcal{H}_{f_\varepsilon} r_{2\varepsilon} \quad f_\varepsilon = (1-\varepsilon)\ \overline{\psi}W \\ r_{1\varepsilon} = d_\varepsilon\ \overline{p_{2\varepsilon}} \\ p_{1\varepsilon} = \overline{r_{2\varepsilon}}\ \overline{d_\varepsilon}^{-1} \end{cases}$$

$d_\varepsilon = \det U'_\varepsilon$ is outer invertible in H^∞ $\forall\varepsilon\in(0,1)$; it may be computed by the relation $|d_\varepsilon|^2=(1-|F|^2)^{-1}$ where $F_\varepsilon = (1-\varepsilon)F$. Now $\overline{r_{1\varepsilon}}\ \overline{d_\varepsilon}^{-1} = p_{2\varepsilon} \in (H^2)^\perp \Rightarrow r_{1\varepsilon}d_\varepsilon{}^{-1}(0) = 0$; being d_ε invertible we have:

$$(2) \qquad r_{1\varepsilon}(0) = 0$$

moreover,

$$(3) \qquad r_{2\varepsilon}(0) = < r_{2\varepsilon}, 1>_{H^2} = \| (\mathbf{I} - \mathcal{A}_\varepsilon{}^*\mathcal{A}_\varepsilon)^{-1/2}\mathbf{1}\|_2$$

Let us state the fundamental result:

<u>Theorem 1.4</u> The operator $\mathcal{A}$ admits a unique minimal symbol $\Leftrightarrow$

$$(4) \qquad \lim_{\varepsilon\to 0} \| (\mathbf{I} - \mathcal{A}_\varepsilon{}^*\mathcal{A}_\varepsilon)^{-1/2}\mathbf{1}\|_2 = +\infty$$

<u>Proof</u>.

It is similar to the proof given in [Sarason; 1985] for Hankel operators, with some slight technical modifications.

Let $\psi\in B(H^\infty)$; $U'_\varepsilon\psi$ is the corresponding symbol of $\mathcal{A}_\varepsilon$ in the sense of the (6) of chapter three. Then:

$$U'_\varepsilon\psi - U'_\varepsilon 0 = [r_{2\varepsilon} (r_{1\varepsilon}\psi + r_{2\varepsilon})]^{-1}\psi d_\varepsilon \in H^\infty$$

The set of values taken by the preceding function in a point z of the unit open disk when ψ varies in $B(H^\infty)$, is a closed disk whose ray is given by:

$$\rho_\varepsilon(z) = |d_\varepsilon(z)| \, [\, | r_{2\varepsilon}(z) |^2 - | r_{1\varepsilon}(z) |^2 \,]$$

We know that $r_{1\varepsilon}r_{2\varepsilon}^{-1} \in B(H^\infty)$; so, from (2) and the Schwarz' lemma, we obtain:

(5) $$|d_\varepsilon(z)| \, | r_{2\varepsilon}(z) |^{-2} \leq \rho_\varepsilon(z) \leq |d_\varepsilon(z)| \, (1 - |z|^2)^{-1} \, | r_{2\varepsilon}(z) |^{-2}$$

Let us note that all the minimal symbols of $\mathcal{A}$ may be obtained by taking limit on sets of the kind $\{U'_\varepsilon\psi \mid \varepsilon \in (0,1)\}$, so a necessary and sufficient condition for the uniqueness of the minimal symbol of $\mathcal{A}$ is that $\rho_\varepsilon(z) \to 0$ when $\varepsilon \to 0$ $\forall z$ in Δ (the preceding limit always exists because $(1 - \varepsilon)^{-1}\rho_\varepsilon(z)$ decreases with ε). From (5) we have that:

$$\lim_{\varepsilon\to 0} \rho_\varepsilon(z) = 0 \quad \forall z \in \Delta \qquad \Leftrightarrow$$

(6) $$\sup_\varepsilon \; |d_\varepsilon(z)|^{-1} \, | r_{2\varepsilon}(z) |^2 = +\infty \quad \forall z \in \Delta$$

Let us observe now that $\{d_\varepsilon^{-1}\}$ is a normal family and $d_\varepsilon^{-1}(z) \neq 0$ $\forall z$ $\forall \varepsilon$. From Hurwitz' theorem (see, for example, [Cartan, 1963], Chap.V, Prop 2.1) there are only two possibilities: either inf $|d_\varepsilon(z)|^{-1} > 0$ $\forall z$ or every limit point of $\{d_\varepsilon^{-1}\}$when $\varepsilon \to 0$ in the open-compact topology is the null function. In the first case, we note that (6) becomes equivalent to:

(7) $$\sup_\varepsilon \; | r_{2\varepsilon}(z) |^2 = +\infty \quad \forall z \in \Delta \, .$$

Using again Hurwitz' theorem we have that (7) is equivalent to:

(8) $$\sup_\varepsilon \; | r_{2\varepsilon}(0) |^2 = +\infty \, .$$

Let us note that $| r_{2\varepsilon}(0) |^2 = \rho_\varepsilon(0)^{-1}|d_\varepsilon(0)|$; $\rho_\varepsilon(0)$ admits a finite limit when $\varepsilon \to 0$, even $|d_\varepsilon(0)|$ admits a finite limit different from zero because of the assumption made on the family $\{d_\varepsilon^{-1}\}$. So necessarily even $|r_{2\varepsilon}(0)|$ admits a limit when $\varepsilon \to 0$; therefore, (8) is equivalent to:

(9) $$\lim_{\varepsilon\to 0} |r_{2\varepsilon}(0)| = +\infty$$

which is exactly (4).

The other possibility is that every limit point of $\{d_\varepsilon^{-1}\}$ is the null function. In this case, necessarily $|F| = 1$ a.e. on the boundary which implies that $\overline{\psi}W = 0$ (owing to $\|\mathcal{A}\| = 1$); the

minimal symbol is thus unique given by ${}^t(0,F)$. On the other hand we have:

$$\| (\mathbf{I} - \mathcal{A}_\varepsilon{}^*\mathcal{A}_\varepsilon)^{-1/2}\mathbf{1}\|_2 = [\varepsilon(2-\varepsilon)]^{-1/2} \to +\infty \text{ when } \varepsilon\to 0$$

and so the proof is complete. Q.E.D.

Obs.2.4 It is easy to see that (4) is equivalent to the two following conditions:

$$\mathbf{1} \notin \mathcal{R}(\mathbf{I} - \mathcal{A}^*\mathcal{A})^{1/2} \tag{10}$$

$$\lim_{\varepsilon\to 0} < (\mathbf{I} - \mathcal{A}_\varepsilon{}^*\mathcal{A}_\varepsilon)^{-1}\mathbf{1}, \mathbf{1}> = +\infty \tag{11}$$

§2 The maximal vector uniqueness criterion

The criterion we now expose is the generalization of a well-known uniqueness criterion for Hankel operators (see, for example, [A.A.K. 1968] and [Sarason; 1967]). It has the defect of not being necessary, but it is simpler to verify than (4) and, moreover, it also gives the structure of the solution.

Def.3.4 Let $T: H \to K$ be a bounded operator acting on Hilbert spaces. A vector $g\in H$, $\|g\|= 1$ is said a **maximal vector** for $T :\Leftrightarrow \|Tg\| = \|T\|$.

Theorem 4.4 Let us suppose that $\mathcal{A}$ has a maximal vector g. Then $\mathcal{A}$ has a unique minimal symbol given by:

$$\begin{bmatrix} g^{-1}\mathcal{H}_{\bar{\psi}w}g \\ F \end{bmatrix} \tag{12}$$

Moreover:

$$| g^{-1}\mathcal{H}_{\bar{\psi}w}g |^2 + |F|^2 = \|\mathcal{A}\|^2 \qquad \text{a.e. on } \partial\Delta. \tag{13}$$

Proof.

Let ${}^t(\phi, F)$ be a minimal symbol for the operator $\mathcal{A}$. Then:

$$\|(\mathcal{H}_{\bar{\psi}W}g, T_Fg)\|_2^2 = \|(\mathcal{H}_\phi g, T_Fg)\|_2^2 = \|\mathcal{H}_\phi g\|_2^2 + \|T_Fg\|_2^2 =$$

$$= \|P_-(\phi g)\|_2^2 + \|Fg\|_2^2 \le \|\phi g\|_2^2 + \|Fg\|_2^2 =$$

$$= \int_0^{2\pi} |\phi|^2|g|^2\frac{d\theta}{2\pi} + \int_0^{2\pi} |F|^2|g|^2\frac{d\theta}{2\pi} = \int_0^{2\pi} (|\phi|^2 + |F|^2)|g|^2\frac{d\theta}{2\pi} \le \| |\phi|^2 + |F|^2 \|_\infty = \|\mathcal{A}\|^2$$

Because g is a maximal vector we have that all the preceding inequalities are in fact equalities; thereby:

$$\phi g \in (H^2)^\perp; \qquad |\phi|^2 + |F|^2 = \|\mathcal{A}\|^2 \text{ a.e.}$$

$$\phi g \in (H^2)^\perp \Rightarrow \mathcal{H}_\phi g = \phi g \Rightarrow \phi = g^{-1}\mathcal{H}_\phi g = g^{-1}\mathcal{H}_{\bar{\psi}W}g. \qquad \text{Q.E.D}$$

Obs.5.4 The relation (13) generalizes the result that the minimal symbol of a Hankel operator having a maximal vector is unimodular.

Obs.6.4 The operator $\mathcal{A}$ admits a maximal vector $\Leftrightarrow$ $\mathcal{A}^*\mathcal{A} = \mathcal{H}_{\bar{\psi}W}{}^*\mathcal{H}_{\bar{\psi}W} + T_F{}^*T_F$ admits a maximal eigenvalue λ. In this case every eigenvector of $\mathcal{A}^*\mathcal{A}$ relative to λ is a maximal vector of $\mathcal{A}$ and vice versa.

Obs.7.4 It follows from the last observation that $\mathcal{A}$ compact $\Rightarrow$ $\mathcal{A}$ has a maximal vector. However $\mathcal{A}$ is compact $\Leftrightarrow$ $\mathcal{H}_{\bar{\psi}W}$ and T_F are compact and it is well-known that T_F is compact $\Leftrightarrow$ $F = 0$. So the operator $\mathcal{A}$ may be compact only in the case it reduces to a purely Hankel operator.

Obs.8.4 It follows from the obs.6.4 that a sufficient condition for the existence of a maximal vector for $\mathcal{A}$ and consequently for the uniqueness of the minimal symbol is that:

$$(14) \qquad \rho_{ess}(\mathcal{A}^*\mathcal{A}) < \|\mathcal{A}^*\mathcal{A}\|$$

where ρ_{ess} is the essential ray of the operator.

So it may be fruitful to analyze the spectrum and the essential spectrum of $\mathcal{A}^*\mathcal{A}$ to verify (14); this has been done in some special cases: in [J.V.; 1986] and, in more generality, in [Z.M. 1987].

CHAPTER FIVE
A parametrization of the minimal symbols.

Let us now suppose that we are in the case of non-uniqueness of the minimal symbols of the operator $\mathcal{A}$ whose norm is supposed to be equal to one. From (7) of chapter four this is equivalent to assuming $\sup \{ |r_{2\varepsilon}(z)| \mid \varepsilon \in (0,1) \} < +\infty \; \forall z \in \Delta$; this implies that $r_{1\varepsilon}$ and $r_{2\varepsilon}$ are uniformly bounded on the compact sets of Δ; so $\exists\, \varepsilon_n \downarrow 0$, $\exists\, r_1, r_2 \in \mathrm{Hol}\,(\Delta)$:

$$r_{1n} := r_{1\varepsilon n} \to r_1$$
$$r_{2n} := r_{2\varepsilon n} \to r_2$$

uniformly on the compact sets of Δ. r_1, r_2 do not necessarily belong to H^∞ but, anyhow:

$$r_{2n}^{-1},\, r_{1n} r_{2n}^{-1} \in B(H^\infty) \quad \Rightarrow \quad r_2^{-1},\, r_1 r_2^{-1} \in B(H^\infty)$$

therefore r_1, r_2 are two holomorphic functions with bounded characteristic and so they have well-defined values on the boundary. It is not restrictive to suppose that $\exists\, \phi_o \in L^\infty$:

$$r_{2n}^{-1}\ \overline{r_{1n}}\ \overline{d_n}^{-1} \to \phi_o$$

in the weak-* topology of H^∞; we have set $d_n = d_{\varepsilon n}$. So (ϕ_o, F) is a minimal symbol of $\mathcal{A}$.

Let now $\psi \in B(H^\infty)$:

$$U'_n\psi - \ r_{2n}^{-1}\ \overline{r_{1n}}\ \overline{d_n}^{-1} = [\, r_{2n}(\, r_{1n}\psi + r_{2n})]^{-1}\psi d_n \in H^\infty \qquad U'_n = U'_{\varepsilon_n}$$

note that:

$$\| U'_n\psi \| \le 1 \quad \| r_{2n}^{-1}\ \overline{r_{1n}}\ \overline{d_n}^{-1} \| \le 1$$

so $[\, r_{2n}(\, r_{1n}\psi + r_{2n})]^{-1}\psi d_n \in H^\infty$ are uniformly bounded by the constant two. Moreover,

$$[\, r_{2n}(\, r_{1n}\psi + r_{2n})]^{-1}\psi d_n \ \to\ [\, r_2(\, r_1\psi + r_2)]^{-1}\psi d$$

pointwise on Δ; d is determined by the condition $|d|^2 = (1-|F|^2)^{-1}$. So we may suppose:

$$[\, r_{2n}(\, r_{1n}\psi + r_{2n})]^{-1}\psi d_n \ \to\ [\, r_2(\, r_1\psi + r_2)]^{-1}\psi d \ \in H^\infty$$

in the weak-* topology of L^∞. This implies that $(\phi_o + \ [\, r_2(\, r_1\psi + r_2)]^{-1}\psi d,\, F)$ is a minimal

symbol of $\mathcal{A}$.

Let us set:

$$U' := \begin{bmatrix} \bar{d}^{-1}\,\bar{r}_1 & \bar{d}^{-1}\,\bar{r}_1 \\ r_1 & r_2 \end{bmatrix}$$

What we will prove is that $U'\psi = \phi_o + [\, r_2(\, r_1\psi + r_2)]^{-1}\psi d$; this is not as trivial a result as might appear on first sight.

Let us start by considering the fuction $\phi_\lambda = \phi_o + [\, r_2(\, r_1\lambda + r_2)]^{-1}\lambda d \quad |\lambda| = 1$

<u>Lemma 1.5</u> $\qquad |\,\phi_\lambda|^2 + |F|^2 = 1 \quad$ a.e.

<u>Proof.</u>

A part from some slight technical differences the proof of this lemma is quite equivalent to the corresponding one in [Sarason; 1985].

$((1-\varepsilon_n)\phi_\lambda\, , F\,)$ is a symbol for $(1-\varepsilon_n)\mathcal{A}$ in $B(H^\infty)$ $\forall n$. So, by theorem 8.3, there exist $\psi_n \in B(H^\infty)$ such that:

(1) $$(1-\varepsilon_n)\phi_\lambda \;=\; U'_n\psi_n \quad \Rightarrow$$

$$(1-\varepsilon_n)\phi_o + (1-\varepsilon_n)[\, r_2(\, r_1\lambda + r_2)]^{-1}\lambda d \;=\; r_{2n}^{\;-1}\;\bar{r}_{1n}\;\;\bar{d}_n^{\;-1} + [\, r_{2n}(\, r_{1n}\psi_n + r_{2n})]^{-1}\psi_n d_n$$

so ψ_n converges to λ pointwise on Δ. On the other hand from (1) we have that:

$$\psi_n \;=\; \frac{r_{2n}(\,(1-\varepsilon_n)\phi_\lambda - \bar{r}_{1n} r_{2n}^{-1}\bar{d}_n^{-1})}{\bar{r}_{2n}\bar{d}_n^{-1}(1 - \phi_\lambda\,(1-\varepsilon_n) r_{1n}\bar{r}_{2n}^{-1}\bar{d}_n)}$$

from which:

$$|\,\psi_n| \le \frac{|\,(1-\varepsilon_n)\phi_\lambda d_n| + |\, r_{1n} r_{2n}^{-1}|}{1 + |\, r_{1n} r_{2n}^{-1}|\,|\,(1-\varepsilon_n)\phi_\lambda d_n|}$$

Note that we have $|\,\phi_\lambda|^2 + |F|^2 \le 1$ a.e. Let us set :

$$E(c) := \{\; \theta \in \partial\Delta \;|\; |\,\phi_\lambda d| \le c\} \qquad c \in (0,1)$$

$$F(n,a) := \{\; \theta \in \partial\Delta \;|\;\; |\, r_{2n}\, d_n^{\;-1}| < a \;\} \quad a>0$$

By the way d is linked to the function F it is clear that our proof is complete if we show that $m(E(c)) = 0\ \forall c \in (0,1)$ where m is the Haar measure on $\partial\Delta$.

On $E(c) \cap F(n,a)$ we have that $|\psi_n| \le [1 + (1-a^{-2})^{1/2}c]^{-1}[c + (1-a^{-2})^{1/2}] = K(c,a)$; from this we derive:

$$|\psi_n(0)| \le 1 - (1 - K(c,a))\, m(E(c) \cap F(n,a))$$

$K(c,a) < 1$; moreover there exists $n_j \to +\infty$ s.t. $|\psi_{nj}(0)| \to 0$. It follows that, for fixed a and c,

$$(2) \qquad m(E(c) \cap F(n_j,a)) \to 0.$$

Now we have that

$$[1 - m(F(n,a))]\log a \le \frac{1}{2\pi}\int_0^{2\pi} \log |r_{2n}(\theta) d_n^{-1}(\theta)|\, d\theta \le M$$

from which we have that $1 - m(F(n,a)) \le M(\log a)^{-1}$, so, by choosing a sufficiently large, we can guarantee that $m(F(n,a))$ is closer to one than any preassigned positive number, for every n. If $\exists c : m(E(c)) > 0$ then $\exists a : 1 - m(F(n,a)) < 1/2\, m(E(c))\ \forall n$, and we would have $m(E(c) \cap F(n,a)) > 1/2\, m(E(c))$ in contradiction with (2). Q.E.D.

Let λ, as above, be a number of unit modulus.

Obs.2.5 We know that $U'_n\lambda \to \phi_\lambda$ in the weak-* topology of L^∞ and consequently, also in the weak topology of L^2. Moreover, we know that:

$$|U'_n\lambda|^2 = 1 - |(1-\varepsilon_n)F|^2 \quad \text{a.e.}$$
$$|\phi_\lambda|^2 = 1 - |F|^2 \quad \text{a.e.}$$

From these two relations, we obtain that $\|U'_n\lambda\|^2 \to \|\phi_\lambda\|^2$. It follows, that $U'_n\lambda \to \phi_\lambda$ in L^2-norm.

Obs.3.5 From what has just been established we easily deduce that

$$\frac{1}{2\pi i}\int_{\partial\Delta} \lambda^{-1} U'_n\lambda\, d\lambda \;\to\; \frac{1}{2\pi i}\int_{\partial\Delta} \lambda^{-1}\phi_\lambda\, d\lambda$$

in L^2-norm and so $r_{2n}^{-1}\ \overline{r}_{1n}\ \overline{d}_n^{-1} \to \phi_o$ in L^2-norm.

Lemma 4.5 $r_{2n}^{-1} \to r_2^{-1}$ in L^2-norm.

Proof.

We have:

$$\frac{1}{2\pi i}\int_0^{2\pi} \lambda^{-2}(U'_n\lambda - U'_n\lambda)\, d\lambda \;\to\; \frac{1}{2\pi i}\int_0^{2\pi} \lambda^{-2}(\phi_\lambda - \phi_o)\, d\lambda$$

in L^2-norm and so $r_{2n}^{-2} d_n \to r_2^{-2} d$ in L^2-norm. We know that $d_n^{-1} \to d^{-1}$ in L^2-norm and so we deduce that $r_{2n}^{-2} \to r_2^{-2}$ in L^1-norm. The last conclusion implies that $\|r_{2n}^{-1}\|_2 \to \| r_2^{-1}\|_2$. On the other hand $r_{2n}^{-1} \to r_2^{-1}$ in the weak topology of L^2. So we conclude that $r_{2n}^{-1} \to r_2^{-1}$ in L^2-norm. Q.E.D.

From obs.3.5 and lemma 4.5, we can assume that, passing to a subsequence if necessary, $r_{2n}^{-1}\ \overline{r_{1n}}\ \overline{d_n}^{-1} \to \phi_o$ a.e on $\partial\Delta$ and $r_{2n}^{-1} \to r_2^{-1}$ a.e on $\partial\Delta$. Moreover, because $U'_n\mathbf{1} \to \phi_1$ in L^2-norm, we have also $U'_n\mathbf{1} \to \phi_1$ a.e on $\partial\Delta$. From this it is easy to derive the following relations:

$$(3) \qquad \begin{cases} r_{1n} \to r_1 & \text{a.e on } \partial\Delta \\ \phi_o = r_2^{-1}\ \overline{r_1}\ \overline{d}^{-1} & \text{a.e on } \partial\Delta \\ U'_n \to U' & \text{a.e on } \partial\Delta \end{cases}$$

Theorem 5.5 $\{(U'\psi, F) \mid \psi \in B(H^\infty)\}$ is the set of all the minimal symbols of $\mathcal{A}$.

Proof.

From (3) we easily obtain that $U'\psi = \phi_o + [\, r_2(\, r_1\psi + r_2)]^{-1}\psi d$ which implies that $(U'\psi,F)$ is a minimal symbol for $\mathcal{A}$, $\forall\ \psi \in B(H^\infty)$.

Now let (ϕ, F) be a minimal symbol for $\mathcal{A}$; we have $|\phi|^2 + |F|^2 \le 1$ a.e. on $\partial\Delta$. $(1-\varepsilon_n)(\phi,F)$ is a symbol for $(1-\varepsilon_n)\mathcal{A}$ in $B(H^\infty)$ so that $\exists\psi_n \in B(H^\infty) : U'_n\psi_n = (1-\varepsilon_n)\,\phi$. We may suppose that $\psi_n \to \psi \in B(H^\infty)$ uniformly on the compact sets of Δ. We have that:

$$(1-\varepsilon_n)\,\phi - r_{2n}^{-1}\ \overline{r_{1n}}\ \overline{d_n}^{-1} = [\, r_{2n}(\, r_{1n}\psi_n + r_{2n})]^{-1}\psi_n d_n \quad \forall n$$

the left side of the preceding relation converges in L^2-norm to $\phi - r_2^{-1}\ \overline{r_1}\ \overline{d}^{-1}$. On the other hand we have that the right side converges in the weak-* topology of L^∞ to $[\, r_2(\, r_1\psi + r_2)]^{-1}\psi d$; therefore we obtain $\phi = r_2^{-1}\ \overline{r_1}\ \overline{d}^{-1} + [\, r_2(\, r_1\psi + r_2)]^{-1}\psi d = U'\psi$ as expected. Q.E.D.

CHAPTER SIX
The finite-dimensional case; an interpolation approach

We want to analyze deeply the optimal problem in the finite-dimensional case that is in the case when ψ is a pure finite Blaschke product.

Let us consider again the optimal problem:

$$(1) \qquad \min_{Z \in H^\infty} \left\| \begin{bmatrix} W - \Psi Z \\ F \end{bmatrix} \right\|_\infty$$

with $\psi = B$ is a finite Blaschke product with simple zeros $\{z_1, \dots z_n\}$ in Δ. $W, F \in H^\infty$ rational functions.

It is well-known that if we set $w_i := W(z_i)$, then $\{W - \psi h \mid h \in H^\infty\}$ is exactly the set of the bounded holomorphic functions interpolating the points (z_i, w_i). So we have that :

$$(2) \quad \varepsilon_o = \min_{Z \in H^\infty} \left\| \begin{bmatrix} W - \Psi Z \\ F \end{bmatrix} \right\|_\infty = \min \{ \|{}^t(f,F)\| \; f \in H^\infty \; f(zi) = wi \}$$

so, as in the case of the finite-dimensional Nehari problem, there is an interpolation problem linked to the original H^∞-optimal problem. A function $f \in H^\infty$ solving problem (2) in the interpolation form is called a minimal interpolating function of (2). We have the following:

<u>Prop.1.6</u> Let us assume that $\varepsilon_0 > \|F\|_\infty$. Then:

(i) there exists a unique minimal interpolation function f which is rational;

(ii) the outer factor g of f is determined by the condition $|g|^2 + |F|^2 = \varepsilon_o^2$ a.e.

(iii) the inner factor of f is a Blaschke product B' of degree at most n-1 which is the minimal solution of the Nevanlinna-Pick interpolation problem relative to the pairs $(z_i, w_i g(z_i)^{-1})$

<u>Proof.</u>

Let us note that the operator $\mathcal{H}_{\bar{B}W}$ is compact because $\bar{B}W \in H^\infty + C(i\mathbb{R})$. So we have :

$$\rho_{ess}(\mathcal{H}_{\bar{B}W}{}^* \mathcal{H}_{\bar{B}W} + T_F{}^* T_F) = \rho_{ess}(T_F{}^* T_F) = \rho_{ess}(T_{|F|^2}) = \| F \|_\infty^2$$

Therefore in the case $\varepsilon_o > \|F\|_\infty$ there exists a maximal vector for $\mathcal{A}$; by applying theorem 4.4, we prove the uniqueness of the solution.

Now, consider the inner-outer factorization of the minimal solution f: f = B'g. From

theorem 4.4 it follows that the outer factor g is determined by the condition $|g|^2 + |F|^2 = \varepsilon_o^2$ a.e..

On the other hand B' is, obviously, a function interpolating the pairs $(z_i,w_i g(z_i)^{-1})$; it has to be the interpolating function of minimal norm because, otherwise, f could not be the minimal solution of the original problem; in particular, this shows that B' is a Blashke product of degree at most n-1.

Finally, f is rational because g and B' are. Q.E.D.

In the case $\varepsilon_o = \|F\|_\infty$ the existence of a maximal vector is not assured anymore and, therefore, we can not carry out the same analysis as before.

Consider the outer function g_ε determined by the condition $|g_\varepsilon|^2+|F|^2= \varepsilon^2$ a.e., where $\varepsilon\geq\|F\|_\infty$. It turns out that g_ε is invertible in H^∞ if and only if $\varepsilon > \|F\|_\infty$. Now, consider the Nevanlinna-Pick interpolation problem (NP_ε) relative to the pairs $(z_i,w_i g_\varepsilon(z_i)^{-1})$; the Hankel operator canonically associated to this problem, when $\varepsilon > \|F\|_\infty$, is $\mathcal{H}_{\bar{B}wg_\varepsilon^{-1}}$. It is easy to see that:

$$\| \mathcal{H}_{\bar{B}wg_\varepsilon^{-1}} \| \leq 1 \quad \Leftrightarrow \quad \varepsilon \geq \varepsilon_o$$

and, if $\varepsilon_o > \|F\|_\infty$, then

$$\| \mathcal{H}_{\bar{B}wg_\varepsilon^{-1}} \| = 1 \quad \Leftrightarrow \quad \varepsilon = \varepsilon_o$$

This observation leads to an algorithm to find the optimal value called the ε-algorithm and illustrated in [C.D.L. 1986]; the main problem connected to the ε-algorithm is that $g_{\|F\|_\infty}$ is not invertible in H^∞ so that, in the case $\varepsilon_o = \|F\|_\infty$, we can not get the optimal value. In the sequel of the paragraph we shall analyze the case $\varepsilon_o = \|F\|_\infty$, showing, in particular, how it is possible to overcome the above difficulty.

Let us consider now the Nevanlinna-Pick interpolation problem relative to the pairs $(z_i,w_i(\varepsilon))$ where as before $i=1,\ldots,n$ and $z_i\neq z_j$ if $i \neq j$ and let us suppose $w_i(\varepsilon) \to 0$ when $\varepsilon\to 0$. Let $c_\varepsilon B_\varepsilon$ be the minimal solution of it; c_ε is a complex constant and B_ε is a finite Blaschke product whose degree is less or equal to n-1.

$$B_\varepsilon(z) = \prod_1^{n-1} \frac{z - v_j(\varepsilon)}{1 - \overline{v_j(\varepsilon)}z} \qquad v_j(\varepsilon) \in \Delta$$

Lemma 2.6 $c_\varepsilon \to 0$ when $\varepsilon \to 0$.

Proof.

We have that $c_\varepsilon B_\varepsilon(z_i) = w_i(\varepsilon) \to 0$. If c_ε does not converge to zero then necessarily $\exists \varepsilon_k \to 0$ such that $B_{\varepsilon k}(z_i) \to 0$ that is:

$$\prod_1^{n-1} \frac{z_i - v_j(\varepsilon)}{1 - \overline{v_j(\varepsilon)} z_i} \to 0 \quad \forall i = 1,\dots,n$$

So we have that $\forall i\ \exists j_i$ such that :

$$\frac{z_i - v_{j_i}(\varepsilon_k)}{1 - \overline{v_{j_i}(\varepsilon_k)} z_i} \to 0$$

Because of $|v_{ji}(\varepsilon_k) z_i| < |z_i| < 1$ we obtain $z_i - v_{ji}(\varepsilon_k) \to 0$ that is $v_{ji}(\varepsilon_k) \to z_i\ \forall i$. Because of $z_i \neq z_l$ if $i \neq l$ then necessarily $j_i \neq j_l$ if $i \neq l$; this is an absurd because the index i takes n distinct values while j at most n-1. Q.E.D.

Let us now return to our initial problem. If $F(z) = \|F\|_\infty\ \forall z \in \Delta$, then the optimal problem (2) is trivial with unique solution given by $f = 0$. Therefore, by the maximum principle, we may assume that F does not assume its maximum value on the open disk Δ. Set $g := g_{\|F\|_\infty}$; we have: $g(z_i) \neq 0\ \forall i$. Therefore it is meaningful to consider the Nevanlinna-Pick interpolation problem (NP) relative to the pairs $(z_i, w_i g(z_i)^{-1})$. Let f' a some interpolating function of (NP); the Hankel operator associated to (NP) is, thus, given by $\mathcal{H}_{\overline{B}f'}$. Moreover let f'_ε the minimal interpolating function of the Nevanlinna-Pick interpolation problems relative to the pairs $(z_i, w_i g(z_i)^{-1} - w_i g_\varepsilon(z_i)^{-1})$; $g_\varepsilon \to g$ uniformly on the compact sets of Δ therefore, by the preceding lemma, we have that $\|f'_\varepsilon\|_\infty \to 0$ (eventually passing to a sequence). If we consider now, the relative Hankel operators, we have:

$$\mathcal{H}_{\overline{B}Wg_\varepsilon^{-1}} - \mathcal{H}_{\overline{B}f'} = \mathcal{H}_{\overline{B}(Wg_\varepsilon^{-1} - f')} = \mathcal{H}_{\overline{B}f'_\varepsilon} \to 0$$

in the operator norm. So $\mathcal{H}_{\overline{B}Wg_\varepsilon^{-1}} \to \mathcal{H}_{\overline{B}f'}$ in the operator norm. We are in the case $\varepsilon_o = \|F\|_\infty$ so, necessarily, $\|\mathcal{H}_{\overline{B}f\varepsilon}\| \leq 1\ \forall \varepsilon$. Therefore, we have $\|\mathcal{H}_{\overline{B}f'}\| \leq 1$.

Theorem.3.6 Suppose $\varepsilon_o=\|F\|_\infty$; then:

(i) $\|\mathcal{H}_{\bar{B}f}\| = 1 \Rightarrow$ there is a unique minimal interpolating function f_o of (2) given by $f_o = B'g$ where B' is the interpolating function of minimal norm relative to (NP).

(ii) $\|\mathcal{H}_{\bar{B}f}\| < 1 \Rightarrow$ there are infinitely many minimal interpolating function of (2) given by $f_\phi = \phi g$ where ϕ is any interpolating function of (NP) whose norm is not greater than 1.

Proof.

Let us note that every function f of the form $f_\phi = \phi g$, where ϕ is an interpolating function of (NP) whose norm is not greater than 1, is a minimal interpolating function of our original problem. Therefore, the proof is complete if we show that every minimal interpolating function is necessarily of this form.

Let $f_o \in H^\infty$ a minimal interpolating function of problem (2) ; clearly $\|g_\varepsilon^{-1} f_o\|_\infty \leq 1$ $\forall \varepsilon > \|F\|_\infty$.Therefore there exists $\varepsilon_k \to \|F\|_\infty : g_{\varepsilon k}^{-1} f_o \to \phi \in B(H^\infty)$ in the compact-open topology; on the other hand $g_{\varepsilon k} \to g$ in the compact-open topology; we conclude that $\phi g = f_o$.

We show now that ϕ is an interpolating function of (NP). Let f' some interpolating function of problem (NP). $\bar{B} g_{\varepsilon k}^{-1} f_o$ are symbols for the Hankel operators $\mathcal{H}_{\bar{B} w g_{\varepsilon k}^{-1}}$; we know that there exist $f'_\varepsilon \in H^\infty$ such that $\bar{B} f'_{\varepsilon k}$ are symbols for $\mathcal{H}_{(\bar{B} w g_{\varepsilon k}^{-1} - \bar{B} f)} : \|f'_\varepsilon\|_\infty \to 0$. Then $\forall k$ $\bar{B} g_{\varepsilon k}^{-1} f_o - \bar{B} f'_{\varepsilon k}$ are symbols of $\mathcal{H}_{\bar{B} f'}$ converging to $\bar{B}\phi$ from which we derive that $\bar{B}\phi$ is a symbol of $\mathcal{H}_{\bar{B} f'}$ and consequently, ϕ is an interpolating function of (NP).

Q.E.D.

Obs.4.6 From the preceding proposition we see that the solution of our initial problem may be unique even if the operator $\mathcal{A}$ does not have a maximal vector; in fact it is quite easy to build an example where this happens.

Obs.5.6 The result contained in theorem 3.6 permits to overcome the difficulty connected to the ε-algorithm; in fact, instead of starting the algorithm from an arbitrary value of ε, now, we can start from $\varepsilon=\|F\|_\infty$ calculating $\|\mathcal{H}_{\bar{B}f}\|$. If $\|\mathcal{H}_{\bar{B}f}\| \leq 1$ then, $\varepsilon_o=\|F\|_\infty$; if $\|\mathcal{H}_{\bar{B}f}\| > 1$ then $\varepsilon_o > \|F\|_\infty$; in the latter case we increase the value of ε and we continue the algorithm.

Acknowledgment:

This research has been conducted under the direction of Professor Sanjoy K. Mitter.

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